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How to experimentally measure the number 5 of the $SO(5)$ theory?

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According to Wilson's theory of critical phenomena, critical exponents are universal functions of d, the dimension of space, and n, the dimension of the symmetry group. $SO(5)$ theory of antiferromagnetism and superconductivity predicts a bicritical point where T_N and T_c intersect. By measuring critical exponents close to the bicritical point, and knowing that $d = 3$, one can experimentally measure the number 5 of the $SO(5)$ theory.

In a system of many strongly interacting degrees of freedom, it is generally hard to make precise quantitative predictions which can be tested experimentally. Theories of high T_c superconductivity generally have to resort to uncontrolled approximations, as a result, it is not possible for experiments to uniquely test the fundamental physical validity of the theory. However, at special values of physical parameters, the basic degrees of freedom may compete so strongly that a new critical point is reached. At this critical point, low energy properties depend only on universal quantities such as the number of space dimension d and the dimension of the symmetry group n, and are independent of the microscopic details of the constituent materials[1,2]. Close to such critical points, a new kind of simplicity and predictibility becomes possible, and the theoretical foundation can be tested unambiguously by experiments.

$SO(5)$ theory predicts a new bicritical point in the two dimensional phase diagram of temperature versus doping where the antiferromagnetic (AF) transition temperature T_N intersects the superconducting (SC) transition temperature T_c. At this critical point, the three component AF order parameter is unified with the two component SC order parameter to form a five component superspin order parameter[3,4]. At this point, the dimension of the symmetry group n is enhanced to 5, and as a result, the critical properties at this point are uniquely different from that of a lower symmetry world. By experimentally tuning into such a bicritical point, and by precisely measuring the critical exponents at this point, experiments can therefore determine the dimension of the symmetry group and test the fundamental validity of the $SO(5)$ theory.

The purpose of this paper is to summarize the various exponents predicted by the $SO(5)$ theory, and to encourage experiments to measure these exponents. Due to chemical complications, the bicritical point has not yet been clearly identified experimentally in the high T_c cuprates. However, such a point does exist in two dimensional organic superconductors which share many physical properties with the high T_c cuprates. Here we also review a insightful theoretical analysis by Murakami and Nagaosa[5], who showed that the the NMR experiments near such a bicritical point measure the dimension of the symmetry group n to be very close to 5.

The model: We start with a generic Ginzburg-Laudau form of the $SO(5)$ model,

$$H = \frac{1}{2}\int d^d\boldsymbol{r}[r_c|\vec{n}|^2 + |\vec{\nabla}\vec{n}|^2 + r_s|\vec{m}|^2 + |\vec{\nabla}\vec{m}|^2 + 2\delta_c|\vec{n}|^4 + 4W|\vec{n}|^2|\vec{m}|^2 + 2\delta_s|\vec{m}|^4]. \quad (1)$$

where $\vec{n}$ and $\vec{m}$ are the order parameters of the SC and the AF respectively. In this paper, we will fix the dimension to $d = 3$ and the expansion parameter $\epsilon = 4 - d = 1$. The mean field phase diagram and RG flows of above effective Hamiltonian have been derived in Ref.[6,7,5]. Defining $F = \delta_c\delta_s - W^2$, we summarize their results in following: **(i)** When $F > 0$, the RG flow converges to the biconical fixed point $(\delta_c, W, \delta_s) = 2\pi^2(0.0905, 0.0847, 0.0536)$ which correponds to *tetracritical* phenomena in mean field phase diagram. In this case, the AF and SC orders can coexist in the low temperature phase; **(ii)** When

$F = 0$, the RG flow converges to a Heisenberg fixed point $\delta_c = \delta_s = W = \frac{2\pi^2}{13}$ which corresponds to *bicritical* behavior. This fixed point has an exact $SO(5)$ rotational symmetry[3]. In this case, there is a direct first order transition between AF and SC. **(iii)** When $F < 0$, the RG flow goes to unstable region $(\delta_c\delta_s) < 0$). The first order transition line between SC and AF branches at a triple critical point and extends until two branches ends at tricritical points. All of above results were discussed and summarized in the schematic diagrams by the authors of Ref.[5]. In the case of $n = 5$, the bicritical and the tetracritical points are very close in parameter space. Starting from a generic point in the parameter space, there is a rapid RG flow towards the bicritical point, followed by a slow flow from the bicritical to the tetracritical point. Therefore, there is a large regime of parameters where the bicritical behavior dominates, and it is possible to observe the $SO(5)$ symmetry. Recent Monte Carlo simulations of the classical $SO(5)$ spin models[8] are consistent with this interpretation of the bicritical point.

Static Exponents: We first discuss the static critical phenomena. At the bicritical point $r_s = r_c = 0$, thermodynamic quantities obey scaling relations. However, $g = r_c - r_s$ is a relevant parameter, and the scaling theory of a bicritical point requires a crossover critical exponent ϕ (Ref.[9]). The scaling postulate for the singular part of the free energy takes the form $(t = |T - T_c(g = 0)|)$,

$$F(t, g, \vec{n}, \vec{m}) = t^{2-\alpha} f(\frac{g}{t^{\phi}}, \frac{\vec{n}}{t^{\beta}}, \frac{\vec{m}}{t^{\beta}}). \tag{2}$$

The exponent β and α take the same value as the isotropic vector model.

The critical exponents α, β, γ, δ, ν and η satisfy the usual scaling relations:

$$\begin{aligned} \alpha &= 2 - d\nu, \quad \beta = \frac{1}{2}(d - 2 + \eta), \\ \gamma &= \nu(2 - \eta), \quad \delta = \frac{d + 2 - \eta}{d - 2 + \eta} \end{aligned} \tag{3}$$

Within second order ϵ expansion, they are given by (see Ref.[1] P.133 for α,β,η ν and δ Ref.[2] P.611 for ϕ):

$$\begin{aligned} \alpha &= -\frac{(n-4)}{2(n+8)}\epsilon - \frac{(n+2)^2(n+28)}{4(n+8)^3}\epsilon^2 \\ \beta &= \frac{1}{2} - \frac{3}{2(n+8)}\epsilon + \frac{(n+2)(2n+1)}{2(n+8)^3}\epsilon^2 \\ \gamma &= 1 + \frac{(n+2)}{2(n+8)}\epsilon + \frac{(n+2)(n^2+22n+52)}{4(n+8)^3}\epsilon^2 \\ \delta &= 3 + \epsilon + \frac{(n^2+14n+60)}{2(n+8)^2}\epsilon^2 \\ \nu &= \frac{1}{2} + \frac{n+2}{4(n+8)}\epsilon + \frac{n+2}{8(n+8)^3}(n^2+23n+60)\epsilon^2 \\ \eta &= \frac{n+2}{2(n+8)^2}\epsilon + \frac{n+2}{8(n+8)^4}(56n+272-n^2)\epsilon^2 \\ \phi &= 1 + \frac{n}{2(n+8)}\epsilon + \frac{n^2+24n+68}{4(n+8)^3}\epsilon^2 \end{aligned} \tag{4}$$

Here we list explicitly the values of the critical exponents in the table 1. It is easy to check the scaling law is approximately satisfied.

Critical temperatures: As already discussed in Ref. [3], the behavior of the SC transition temperature T_c and the AF transition temperature T_N close to the bicritical point are governed by the $SO(5)$ bicritical exponent ϕ. In the neighborhood of the bi-critical point, divergent quantities generally behave like:

$$\chi(T, g) \sim t^{-\gamma_5} X(g/t^{\phi}) \tag{5}$$

where $t = T - T_c(g = 0)$ and $X(z)$ is a scaling function, normalized such that $X(0) = 1$. However, unlike the usual scaling functions, $X(z)$ diverges at two points $z_2 > 0$ and $z_3 < 0$:

$$X(z) \sim (z - z_2)^{-\gamma_2} \quad ; \quad X(z) \sim (z - z_3)^{-\gamma_3} \tag{6}$$

Therefore, sufficiently close to the $SO(5)$ bicritical point, $g/t^{\phi} \ll 1$, and the critical behavior is given by the new $SO(5)$ exponent γ_5. Away from the $SO(5)$ bicritical point, the divergence of physical quantities are determined by the divergence of $X(z)$. For $g > 0$, the critical temperature is given by $g/t^{\phi} = z_2$, or

$$T_c(g) = T_c(0) + A g^{1/\phi} \tag{7}$$

where A is a constant. Similar arguments applies for the case of $g < 0$. This way, by measuring

the precise values of both T_c and T_N close to the bicritical point, one can determine the value of the crossover exponent ϕ and compare it with the $SO(5)$ prediction of $\phi = 1.314$.

London penetration length: Near the superconducting to normal phase transition, this quantity scales like[10]

$$\lambda \sim \rho_s^{-\frac{1}{2}} \sim \xi^{-(2-D)/2} \sim t^{-\nu/2} \quad (8)$$

This is a very interesting quantity, since its critical behavior has already been measured by Kamal et al. [11] for YBCO superconductors, and the exponent was found to be consistent with the XY value of $\nu_2 = 0.655$. Here we suggest to measure the critical behavior of λ for doping levels ranging from optimal to deeply underdoped regime. The $SO(5)$ theory predicts that data for all doping levels x can be fit into a single scaling curve:

$$\chi(T,x) \sim t^{-\nu_5/2} Y(x/t^{\phi}) \quad (9)$$

where $Y(0) = 1$ and it diverges near $z_2 > 0$ as:

$$Y(z) \sim (z - z_2)^{-\nu_2/2} \quad (10)$$

If one can get sufficiently close to the bicritical regime, one can determine both ν_5 and ϕ and compare with the $SO(5)$ predictions of $\nu_5 = 0.714$ and $\phi = 1.314$. Together with these precisely predicted values, the fitting into a single scaling curve for all doping levels provides a highly non-trivial quantitative test of the $SO(5)$ theory.

Other static quantities should all follow similar scaling relations close to the bicritical point.

Dynamic exponent: Under the dynamic scaling hypothesis, the typical frequency or the relaxation rate ω scales as $\omega \sim \xi^{-z}$. Standard arguments in dynamical critical phenomena gives $z = d/2$ generally, and $z = \phi/\nu$ near a bicritical point[12].

Nuclear magnetic relaxation rate: $1/T_1$ is given by the following response function:

$$1/T_1 = lim_{\omega \to 0} \frac{1}{\omega} \int \frac{d^d k}{(2\pi)^d} \chi(k,\omega) \quad (11)$$

where $\chi(k,\omega)$ is the *imaginary part* of the spin response function, which near a critical point behaves like

$$\chi(k,\omega) = \xi^{2-\eta} Y(\bar{k},\bar{\omega}) \quad (12)$$

Here $Y(\bar{k},\bar{\omega})$ is a scaled, dimensionless spin correlation function of the dimensionless variable $\bar{k} = k\xi$ and $\bar{\omega} = \omega\xi^z$. Expressed in terms of the rescaled variables,

$$1/T_1 = \xi^{z-d+2-\eta} lim_{\bar{\omega} \to 0} \frac{1}{\bar{\omega}} \int \frac{d^d \bar{k}}{(2\pi)^d} Y(\bar{k},\bar{\omega}) \quad (13)$$

from which we can see that the scaling behavior of $1/T_1$ is given by:

$$1/T_1 = \xi^{z-d+2-\eta} = t^{-x} \quad (14)$$

where $x = \nu(z - 1 - \eta)$. Applying the results of the ϵ expansion listed in the previous table, we obtain $x = \nu(z - 1 - \eta) = 0.573$ close to a $SO(5)$ bicritical point. For a regular antiferromagnetic transition of the $O(3)$ symmetry class, we obtain $x = \nu(z - 1 - \eta) = 0.67(1.5 - 1 - 0.039) = 0.312$.

Frequency-dependent conductivity: In the superconducting state $T < T_c$, and for low frequency, the complex conductivity takes the form

$$\sigma(\omega) \sim \frac{\rho_s}{-i\omega} \quad (15)$$

In the critical region, as we pointed out before, $\rho_s \sim \xi^{2-d}$, therefore, the dynamic conductivity scales as

$$\sigma(\omega) \sim \xi^{2-d}/\omega \sim \omega^{-\frac{z+2-d}{z}} \quad (16)$$

at $T = T_c$ for low frequency. At a $SO(5)$ biciritical point, the exponent is given by $\frac{z+2-d}{z} = 0.46$ compared with $\frac{z+2-d}{z} = 0.33$ for a ordinary superconductor to normal transition in the XY universality class.

Experimental status: Possibly due to chemical complications, it is hard to reach a uniform state in the deeply underdoped regime of the high T_c superconductors. For this reason, the existence of a bicritical in the high T_c superconductors has neither been discovered nor refuted. One of the greatest experimental challenge in this field is to prepare better and more uniform materials in the deeply underdoped regime. Given such materials, it is most feasible to measure the critical properties of the London penetration length by microwave cavity experiment. As clearly demonstrated in this work, such system can provide definite and quantitative test of the $SO(5)$ theory of high T_c superconductivity.

Encouraging experimental evidence for a $SO(5)$ bicritical point does exist in a class of 2D organic superconductors called *bedt* salt. These material share most common physical properties with the cuprates, and a AF to SC transition can be induced by pressure. A bicritical point exists where T_c and T_N intersect each other. Kanoda and coworkers[13] measured the $1/T_1$ rate both in the AF region and the bicritical region. Murakami and Nagaosa[5] analyzed the experimental data. The $1/T_1$ exponent in the AF region was measured to be $x_{AF} = 0.30$, compared with the theoretical prediction of $x_3 = 0.312$. In the bicritical region, the experimental fit gives $x_{bi} = 0.56$, compared with the $SO(5)$ theoretical prediction of $x_5 = 0.573$. This is the first experiment which directly measures the dimension of the symmetry group close to a AF/SC bicritical point, and determines n to be close to 5.

Conclusions: $SO(5)$ theory makes precise and quantitative predictions on the critical exponents near a bicritical point. We strongly encourage experiments to be carried out in deeply underdoped regime of high T_c superconductors and to look for the bicritical point. Measurement of the critical exponents associated with various physical quantities can uniquely test the fundamental validity of the $SO(5)$ theory, and can measure the number 5 of the the $SO(5)$ theory in a direct and unambiguous fashion.

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n	2	3	4	5
α	-0.02	-0.1	-0.167	-0.222
β	0.36	0.377	0.391	0.402
γ	1.3	1.347	1.385	1.418
δ	4.46	4.559	4.458	4.458
ν	0.655	0.678	0.698	0.714
η	0.039	0.039	0.038	0.037
ϕ	1.16	1.22	1.27	1.314
z	1.771	1.799	1.819	1.840

Table 1
Static and dynamic exponents for n vector model

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